The Relative Velocity between Two Confocal Keplerian Orbits

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Abstract

We derive analytically the relative velocity vector and relative velocity magnitude between two particles on confocal Keplerian orbits.

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1 Introduction

Hilton et al. 1996 [1] and Hilton 2002 [2] used a two-body scattering angle as a filter for determining significant asteroid-asteroid close approaches. The scattering angle φ in the center-of-mass frame is given by

$$\tan\frac{\varphi}{2} = \frac{G(m_1 + m_2)}{v^2 b} \tag{1}$$

where m_1 and m_2 are the asteroid masses, b is the impact parameter, and v is the magnitude of the relative encounter velocity. Thus, the relative encounter speed is important to the observed effects of an encounter deflection. It is possible for a distant encounter at low relative speed to suffer a larger deflection than a closer encounter at higher relative speed. In this Note, we calculate explicitly the relative encounter velocity and its magnitude for two confocal Keplerian orbits, in terms of relative orbital elements $\{\alpha, e_1, e_2, \iota, \omega, \Omega; E_1, E_2\}$. First, however, we present the relative elements in terms of the orbital elements of the two orbits with respect to some intertial frame, $\{a_k, e_k, \iota_k, \omega_k, \Omega_k, M_k\},\ k = 1, 2$.

2 Relative Elements and the Relative Distance Vector

Consider the set of relative orbital elements $\{\alpha, e_1, e_2, \iota, \omega, \Omega; E_1, E_2\}$, where

$$\alpha = \frac{a_2}{a_1}\ , \tag{2}$$

the relative orientation elements $\{\iota, \omega, \Omega\}$ are as shown in Figure 1, and $E_1(t)$ and $E_2(t)$ are the eccentric anomalies of the two bodies. The transformation from separate orbital elements for each orbit to relative elements is given by (Murison and Munteanu 2007 [3])

$$\left.\begin{array}{rcl}
\cos\Omega\,\sin\iota &=& \sin\omega_1\sin\iota_2\sin\Delta\Omega - \cos\omega_1(\sin\iota_1\cos\iota_2 - \cos\iota_1\sin\iota_2\cos\Delta\Omega) \\
\sin\Omega\,\sin\iota &=& \cos\omega_1\sin\iota_2\sin\Delta\Omega + \sin\omega_1(\sin\iota_1\cos\iota_2 - \cos\iota_1\sin\iota_2\cos\Delta\Omega) \\
\cos\omega\,\sin\iota &=& \sin\omega_2\sin\iota_1\sin\Delta\Omega + \cos\omega_2(\cos\iota_1\sin\iota_2 - \sin\iota_1\cos\iota_2\cos\Delta\Omega) \\
\sin\omega\,\sin\iota &=& -\cos\omega_2\sin\iota_1\sin\Delta\Omega + \sin\omega_2(\cos\iota_1\sin\iota_2 - \sin\iota_1\cos\iota_2\cos\Delta\Omega) \\
\cos\iota &=& \cos\iota_1\cos\iota_2 + \sin\iota_1\sin\iota_2\cos\Delta\Omega
\end{array}\right\} \tag{3}$$

where $\Delta\Omega = \Omega_2 - \Omega_1$. The eccentric anomalies can be calculated from the respective mean anomalies via the Kepler equation,

$$E(t) - e\sin E(t) = M(t) = n(t - \tau) = M_0 + n(t - t_0) \tag{4}$$

where M_0 is the mean anomaly at epoch $t = t_0$, τ is a time of pericenter passage, and n is the mean motion $2\pi/T$ with T the orbital period. With equations (3) and (4) we may transform from separate orbital elements with respect to some inertial frame at epoch t_0 to relative elements.

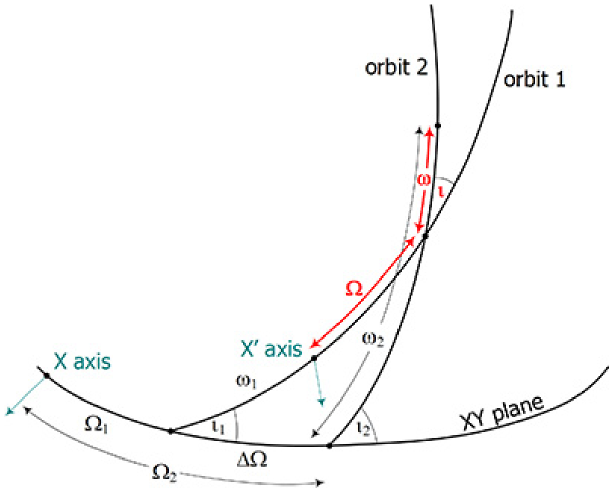

Figure 1: Relative orbital elements.

In terms of the relative elements, and with respect to a coordinate frame whose x axis is along the pericenter and z axis is parallel to the angular momentum vector of orbit 1, the relative distance vector is

$$\frac{1}{a_1}\overrightarrow{R}(t) = \frac{1}{a_1}(\overrightarrow{r}_2 - \overrightarrow{r}_1) = \alpha \mathcal{Q}\left(\Omega, \iota, \omega\right) \cdot \overrightarrow{q}\left(E_2\right) - \overrightarrow{q}\left(E_1\right) \tag{5}$$

where in general a Keplerian orbit may be expressed in the form $\overrightarrow{r} = a\mathcal{Q}\left(\Omega, \iota, \omega\right) \cdot \overrightarrow{q}\left(E\right)$, where the vector

$$\overrightarrow{q}\left(E\right) = \begin{bmatrix} \cos E - e \\ \sqrt{1-e^2}\sin E \\ 0 \end{bmatrix} \tag{6}$$

specifies the position along the orbit path, and the orbit orientation matrix $\mathcal{Q}\left(\Omega, \iota, \omega\right)$ is a product of three orthogonal rotations,

$$\mathcal{Q}\left(\Omega, \iota, \omega\right) = \begin{bmatrix} \cos\Omega\,\cos\omega - \sin\Omega\,\sin\omega\,\cos\iota & -\cos\Omega\,\sin\omega - \sin\Omega\,\cos\omega\,\cos\iota & \sin\Omega\,\sin\iota \\ \sin\Omega\,\cos\omega + \cos\Omega\,\sin\omega\,\cos\iota & -\sin\Omega\,\sin\omega\, + \cos\Omega\,\cos\omega\,\cos\iota & -\cos\Omega\,\sin\iota \\ \sin\omega\,\sin\iota & \cos\omega\,\sin\iota & \cos\iota \end{bmatrix} \tag{7}$$

This formulation cleanly separates the orbit scale a, the shape of the orbit e plus the position of the particle at a given time $\overrightarrow{q}$, and the orientation of the orbit in space $\mathcal{Q}\left(\Omega, \iota, \omega\right)$.

3 Relative Velocity

Differentiation of the relative distance vector yields the relative velocity vector,

$$\frac{1}{a_1}\overrightarrow{V}(t) = \frac{1}{a_1}\frac{d\overrightarrow{R}}{dt} = \alpha \mathcal{Q}(\Omega, \iota, \omega) \cdot \frac{d\overrightarrow{q}(E_2(t))}{dt} - \frac{d\overrightarrow{q}(E_1(t))}{dt} \tag{8}$$

Now, from the Kepler equation (4), we have

$$\dot{E} = \frac{n}{1 - e\cos E} \tag{9}$$

Define a mass parameter

$$\mu_k \equiv G(M + m_k), \quad k \in \{1, 2\} \tag{10}$$

where M is the central mass. Since $M \gg m_k$,

$$\frac{\mu_2}{\mu_1} = \frac{M + m_2}{M + m_1} = \frac{1 + \varepsilon_2}{1 + \varepsilon_1} = 1 + \varepsilon_2 - \varepsilon_1 - \varepsilon_1\varepsilon_2 + \varepsilon_1^2 - \ldots \tag{11}$$

where $\varepsilon_k = \frac{m_k}{M}$. Thus, for the asteroids $\frac{\mu_2}{\mu_1} \approx 1$ and $\mu_1 \approx \mu_2 \approx GM_\odot \equiv \mu_\odot$. The mean motion and semimajor axis are related by Kepler's third law, $\mu = n^2a^3$ (where, in general, μ is G times the sum of the two masses). Hence, we may generally write

$$\frac{d\overrightarrow{q}}{dt} = \frac{\partial\overrightarrow{q}}{\partial E}\dot{E} = \frac{n}{1 - e\cos E}\frac{\partial\overrightarrow{q}}{\partial E} = \frac{1}{1 - e\cos E}\sqrt{\frac{\mu}{a^3}}\begin{bmatrix} -\sin E \\ \sqrt{1 - e^2}\cos E \\ 0 \end{bmatrix} \tag{12}$$

Let $\beta = \alpha - 1 = \frac{a_2 - a_1}{a_1}$. Then the relative velocity, (8), becomes

$$\sqrt{\frac{a_1}{\mu_\odot}}\overrightarrow{V}(t) = \sqrt{\frac{1}{1+\beta}}\frac{1}{1 - e_2\cos E_2}\mathcal{Q}(\Omega, \iota, \omega) \cdot \frac{\partial\overrightarrow{q}(E_2)}{\partial E_2} - \frac{1}{1 - e_1\cos E_1}\frac{\partial\overrightarrow{q}(E_1)}{\partial E_1} \tag{13}$$

where the velocity is scaled by the velocity of a circular orbit of radius a_1,

$$n_1a_1 = \sqrt{\frac{\mu_\odot}{a_1}} \tag{14}$$

In component form, eq. (13) is

$$\begin{aligned} \sqrt{\tfrac{a_1}{\mu_\odot}}\overrightarrow{V}(t) &= \tfrac{1}{1 - e_2\cos E_2}\tfrac{\overrightarrow{B}(E_1, E_2)}{\sqrt{1+\beta}} - \tfrac{1}{1 - e_1\cos E_1}\begin{bmatrix} -\sin E_1 \\ \sqrt{1 - e_1^2}\cos E_1 \\ 0 \end{bmatrix} \\ &= \tfrac{1 - \frac{1}{2}\beta + \frac{3}{8}\beta^2 - \cdots}{1 - e_2\cos E_2}\overrightarrow{B}(E_1, E_2) - \tfrac{1}{1 - e_1\cos E_1}\begin{bmatrix} -\sin E_1 \\ \sqrt{1 - e_1^2}\cos E_1 \\ 0 \end{bmatrix} \end{aligned} \tag{15}$$

where

$$\overrightarrow{B}(E_1,E_2)=\begin{bmatrix} C_2\sqrt{1-e_2^2}\cos E_2-C_4\sin E_2 \\ C_1\sqrt{1-e_2^2}\cos E_2-C_3\sin E_2 \\ \Big(\cos\omega\sqrt{1-e_2^2}\cos E_2-\sin\omega\sin E_2\Big)\sin\iota \end{bmatrix} \tag{16}$$

and

$$\left.\begin{array}{rcr} C_1 & = & -\sin\Omega\,\sin\omega+\cos\Omega\,\cos\omega\,\cos\iota \\ C_2 & = & -\cos\Omega\,\sin\omega-\sin\Omega\,\cos\omega\,\cos\iota \\ C_3 & = & \sin\Omega\,\cos\omega+\cos\Omega\,\sin\omega\,\cos\iota \\ C_4 & = & \cos\Omega\,\cos\omega-\sin\Omega\,\sin\omega\,\cos\iota \end{array}\right\} \tag{17}$$

Upon squaring (15), we find the relative velocity magnitude,

$$\frac{a_1}{\mu_\odot}\left\|\overrightarrow{V}(t)\right\|^2=\frac{1+e_1\cos E_1}{1-e_1\cos E_1}+\frac{1}{1+\beta}\frac{1+e_2\cos E_2}{1-e_2\cos E_2}-\frac{2}{\sqrt{1+\beta}}\frac{g(E_1,E_2)}{(1-e_1\cos E_1)(1-e_2\cos E_2)} \tag{18}$$

where

$$\begin{aligned} g(E_1,E_2) \quad = \quad & \sqrt{1-e_2^2}\cos E_2\Big(C_1\sqrt{1-e_1^2}\cos E_1-C_2\sin E_1\Big) \\ & -\sin E_2\Big(C_3\sqrt{1-e_1^2}\cos E_1-C_4\sin E_1\Big) \end{aligned} \tag{19}$$

An optimized C function to calculate eq. (18) is shown in the Appendix (Section 4). For completeness, expanding on ε, we have

$$\begin{aligned} \tfrac{a_1}{\mu_\odot}\left\|\overrightarrow{V}(t)\right\|^2 \quad = \quad & \tfrac{1+e_1\cos E_1}{1-e_1\cos E_1}+\tfrac{1+e_2\cos E_2}{1-e_2\cos E_2}\big(1-\beta+\beta^2-\cdots\big) \\ & -\tfrac{2g(E_1,E_2)}{(1-e_1\cos E_1)(1-e_2\cos E_2)}\big(1-\tfrac{1}{2}\beta+\tfrac{3}{8}\beta^2-\cdots\big) \end{aligned} \tag{20}$$

References

[1] Hilton, J.L., Seidelmann, P.K., and Middour, J. (1996), “Prospects for Determining Asteroid Masses”, *Astronomical Journal* **112**, 2319-2329.

[2] Hilton, J.L. (2002), “Asteroid Masses and Densities”, in *Asteroids III*, eds. Bottke et al., University of Arizona Press, Tuscon, pp. 103-112.

[3] Murison, M.A., and Munteanu, A. (2007), “On the Distance Function between Two Confocal Keplerian Orbits”, submitted to *Astronomical Journal.*

4 Appendix: Optimized C Code for the Squared Relative Velocity

```
double vsquared( double E1, double E2, double e1, double e2,
                 double beta, double iota, double peri, double node )
{
    double t6  = sin(node);
    double t7  = cos(peri);
    double t19 = t6*t7;
    double t5  = cos(node);
    double t18 = t5*t7;
    double t8  = sin(peri);
    double t17 = t6*t8;
    double t16 = t5*t8;
    double t11 = cos(E2);
    double t1  = e2*e2;
    double t20 = sqrt(1.0-t1);
    double t15 = t20*t11;
    double t14 = e2*t11;
    double t12 = cos(E1);
    double t13 = e1*t12;
    double t10 = sin(E2);
    double t9  = cos(iota);
    double t4  = 1.0+beta;
    double t2  = 1.0/(t13-1.0);
    double t24 = sqrt(t4);
    double t34 = e1*e1;
    double t36 = sqrt(1.0-t34);
    double t45 = sin(E1);
    return ( (-t13-1.0)*t2
             + ( -2.0/t24*( t12*((-t17+t9*t18)*t15-(t19+t9*t16)*t10)*t36
                          + ((t9*t19+t16)*t15-(-t18+t9*t17)*t10)*t45 )*t2
                 + (-t14-1.0)/t4 )/(-1.0+t14) );
}
```