

# Associated Legendre Polynomials in $\sin \phi$ to Twelfth Order

**Marc A. Murison**

*Astronomical Applications Department  
U.S. Naval Observatory  
Washington, DC*

murison@riemann.usno.navy.mil  
<http://aa.usno.navy.mil/murison/>

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Here is a function that generates the associated Legendre function

$L_{n,m} = (1-x^2)^{\binom{m}{2}} \text{diff}(P_n(x), x \$ m)$  of an argument  $\phi$ , where  $P_n(x)$  is the Legendre polynomial of order  $n$ :

*restart*

```
ALegendre := proc( n, m, phi )
  local A, x;
  if m > 0 then
    A := (1-x^2)^(m/2)*diff( orthopoly[P](n,x), x$m );
  else
    A := orthopoly[P](n,x);
  fi;
  factor(subs(x=phi,A));
end;
```

For example,

`ALegendre(5, 3,  $\phi$ )`

$$\frac{105}{2} (-(\phi - 1)(\phi + 1))^{(3/2)} (3\phi - 1)(3\phi + 1)$$

We can convert the expressions into C functions by first converting into a Maple procedure then transforming into C. For example,

`with(codegen)`

`func := makeproc([optimize(%%, tryhard)])`

`func := proc()`

`local t1, t2;`

`t2 := 3* $\phi$ ; t1 := 105 / 2*(-( $\phi$  - 1)*( $\phi$  + 1))^(3 / 2)*(t2 - 1)*(t2 + 1)`

**end proc**

**C(func)**

```
#include <math.h>
double func()
{
    double t1;
    double t2;
    {
        t2 = 3.0*phi;
        t1 = 105.0/2.0*sqrt(-pow(phi-1.0,3.0)*pow(phi+1.0,3.0))*(t2-1.0)*(t2+1.0);
        return(t1);
    }
}
```

Write a procedure to automate this for all nonzero associated Legendre polynomials up to a given order and mode, and write the C functions to disk.

```
MakePolys := proc( nmax::nonnegint, mmax::nonnegint, F )
    local n, m, poly, prog;
    poly := [];
    for n from 1 to nmax do
        for m from 0 to mmax do
            poly := ALegendre(n,m,sin(phi));
            if poly <> 0 then
                prog := makeproc([codegen[optimize](poly,tryhard)]);
                codegen[C]( prog, filename=cat("M:/j1h_",n,"_",m,".dat") );
                printf( "n=%d, m=%d\n", n, m );
                print( poly );
                printf("\n\n");
            fi;
        od;
    od;
    NULL;
end proc;
```

Now generate all nonzero associated Legendre polynomials to order and mode 12 for argument  $\sin \phi$ .

**MakePolys( 12, 12, sin( $\phi$ ))**

n=1, m=0

$$\sin(\phi)$$

n=1, m=1

$$\sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)}$$

n=2, m=0

$$\frac{3}{2}\sin(\phi)^2 - \frac{1}{2}$$

n=2, m=1

$$3 \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} \sin(\phi)$$

$$n=2, m=2$$

$$-3 (\sin(\phi) - 1) (\sin(\phi) + 1)$$

$$n=3, m=0$$

$$\frac{1}{2} \sin(\phi) (5 \sin(\phi)^2 - 3)$$

$$n=3, m=1$$

$$\frac{3}{2} \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} (5 \sin(\phi)^2 - 1)$$

$$n=3, m=2$$

$$-15 (\sin(\phi) - 1) (\sin(\phi) + 1) \sin(\phi)$$

$$n=3, m=3$$

$$15 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)}$$

$$n=4, m=0$$

$$\frac{35}{8} \sin(\phi)^4 - \frac{15}{4} \sin(\phi)^2 + \frac{3}{8}$$

$$n=4, m=1$$

$$\frac{5}{2} \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} \sin(\phi) (7 \sin(\phi)^2 - 3)$$

$$n=4, m=2$$

$$-\frac{15}{2} (\sin(\phi) - 1) (\sin(\phi) + 1) (7 \sin(\phi)^2 - 1)$$

$$n=4, m=3$$

$$105 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} \sin(\phi)$$

$$n=4, m=4$$

$$105 (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2$$

$$n=5, m=0$$

$$\frac{1}{8} \sin(\phi) (63 \sin(\phi)^4 - 70 \sin(\phi)^2 + 15)$$

$$n=5, m=1$$

$$\frac{15}{8} \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} (21 \sin(\phi)^4 - 14 \sin(\phi)^2 + 1)$$

$$n=5, m=2$$

$$-\frac{105}{2} (\sin(\phi) - 1)(\sin(\phi) + 1) \sin(\phi) (3 \sin(\phi)^2 - 1)$$

$$n=5, m=3$$

$$\frac{105}{2} (-(\sin(\phi) - 1)(\sin(\phi) + 1))^{(3/2)} (3 \sin(\phi) - 1)(3 \sin(\phi) + 1)$$

$$n=5, m=4$$

$$945 (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 \sin(\phi)$$

$$n=5, m=5$$

$$945 (-(\sin(\phi) - 1)(\sin(\phi) + 1))^{(5/2)}$$

$$n=6, m=0$$

$$\frac{231}{16} \sin(\phi)^6 - \frac{315}{16} \sin(\phi)^4 + \frac{105}{16} \sin(\phi)^2 - \frac{5}{16}$$

$$n=6, m=1$$

$$\frac{21}{8} \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} \sin(\phi) (33 \sin(\phi)^4 - 30 \sin(\phi)^2 + 5)$$

$$n=6, m=2$$

$$-\frac{105}{8} (\sin(\phi) - 1) (\sin(\phi) + 1) (33 \sin(\phi)^4 - 18 \sin(\phi)^2 + 1)$$

$$n=6, m=3$$

$$\frac{315}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} \sin(\phi) (11 \sin(\phi)^2 - 3)$$

$$n=6, m=4$$

$$\frac{945}{2} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 (11 \sin(\phi)^2 - 1)$$

$$n=6, m=5$$

$$10395 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} \sin(\phi)$$

$$n=6, m=6$$

$$-10395 (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3$$

$$n=7, m=0$$

$$\frac{1}{16} \sin(\phi) (429 \sin(\phi)^6 - 693 \sin(\phi)^4 + 315 \sin(\phi)^2 - 35)$$

$$n=7, m=1$$

$$\frac{7}{16} \sqrt{-(\sin(\phi) - 1) (\sin(\phi) + 1) (429 \sin(\phi)^6 - 495 \sin(\phi)^4 + 135 \sin(\phi)^2 - 5)}$$

$$n=7, m=2$$

$$-\frac{63}{8} (\sin(\phi) - 1) (\sin(\phi) + 1) \sin(\phi) (143 \sin(\phi)^4 - 110 \sin(\phi)^2 + 15)$$

$$n=7, m=3$$

$$\frac{315}{8} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} (143 \sin(\phi)^4 - 66 \sin(\phi)^2 + 3)$$

$$n=7, m=4$$

$$\frac{3465}{2} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 \sin(\phi) (13 \sin(\phi)^2 - 3)$$

$$n=7, m=5$$

$$\frac{10395}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} (13 \sin(\phi)^2 - 1)$$

$$n=7, m=6$$

$$-135135 (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 \sin(\phi)$$

$$n=7, m=7$$

$$135135 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(7/2)}$$

$$n=8, m=0$$

$$\frac{6435}{128} \sin(\phi)^8 - \frac{3003}{32} \sin(\phi)^6 + \frac{3465}{64} \sin(\phi)^4 - \frac{315}{32} \sin(\phi)^2 + \frac{35}{128}$$

$$n=8, m=1$$

$$\frac{9}{16} \sqrt{-(\sin(\phi) - 1) (\sin(\phi) + 1) \sin(\phi) (715 \sin(\phi)^6 - 1001 \sin(\phi)^4 + 385 \sin(\phi)^2 - 35)}$$

$$n=8, m=2$$

$$-\frac{315}{16} (\sin(\phi) - 1) (\sin(\phi) + 1) (143 \sin(\phi)^6 - 143 \sin(\phi)^4 + 33 \sin(\phi)^2 - 1)$$

$$n=8, m=3$$

$$\frac{3465}{8} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} \sin(\phi) (39 \sin(\phi)^4 - 26 \sin(\phi)^2 + 3)$$

$$n=8, m=4$$

$$\frac{10395}{8} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 (65 \sin(\phi)^4 - 26 \sin(\phi)^2 + 1)$$

$$n=8, m=5$$

$$\frac{135135}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} \sin(\phi) (5 \sin(\phi)^2 - 1)$$

$$n=8, m=6$$

$$-\frac{135135}{2} (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 (15 \sin(\phi)^2 - 1)$$

$$n=8, m=7$$

$$2027025 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(7/2)} \sin(\phi)$$

$$n=8, m=8$$

$$2027025 (\sin(\phi) - 1)^4 (\sin(\phi) + 1)^4$$

$$n=9, m=0$$

$$\frac{1}{128} \sin(\phi) (12155 \sin(\phi)^8 - 25740 \sin(\phi)^6 + 18018 \sin(\phi)^4 - 4620 \sin(\phi)^2 + 315)$$

$$n=9, m=1$$

$$\frac{45}{128} \sqrt{-(\sin(\phi) - 1) (\sin(\phi) + 1)} (2431 \sin(\phi)^8 - 4004 \sin(\phi)^6 + 2002 \sin(\phi)^4 - 308 \sin(\phi)^2 + 7)$$

$$n=9, m=2$$

$$-\frac{495}{16} (\sin(\phi) - 1) (\sin(\phi) + 1) \sin(\phi) (221 \sin(\phi)^6 - 273 \sin(\phi)^4 + 91 \sin(\phi)^2 - 7)$$

$$n=9, m=3$$

$$\frac{3465}{16} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} (221 \sin(\phi)^6 - 195 \sin(\phi)^4 + 39 \sin(\phi)^2 - 1)$$

$$n=9, m=4$$

$$\frac{135135}{8} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 \sin(\phi) (17 \sin(\phi)^4 - 10 \sin(\phi)^2 + 1)$$

n=9, m=5

$$\frac{135135}{8} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} (85 \sin(\phi)^4 - 30 \sin(\phi)^2 + 1)$$

n=9, m=6

$$-\frac{675675}{2} (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 \sin(\phi) (17 \sin(\phi)^2 - 3)$$

n=9, m=7

$$\frac{2027025}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(7/2)} (17 \sin(\phi)^2 - 1)$$

n=9, m=8

$$34459425 (\sin(\phi) - 1)^4 (\sin(\phi) + 1)^4 \sin(\phi)$$

n=9, m=9

$$34459425 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(9/2)}$$

n=10, m=0

$$\frac{46189}{256} \sin(\phi)^{10} - \frac{109395}{256} \sin(\phi)^8 + \frac{45045}{128} \sin(\phi)^6 - \frac{15015}{128} \sin(\phi)^4 + \frac{3465}{256} \sin(\phi)^2 - \frac{63}{256}$$

n=10, m=1

$$\frac{55}{128} \sqrt{-\sin(\phi) (\sin(\phi) - 1) (\sin(\phi) + 1)} (4199 \sin(\phi)^8 - 7956 \sin(\phi)^6 + 4914 \sin(\phi)^4 - 1092 \sin(\phi)^2 + 63)$$

n=10, m=2

$$-\frac{495}{128} (\sin(\phi) - 1) (\sin(\phi) + 1) (4199 \sin(\phi)^8 - 6188 \sin(\phi)^6 + 2730 \sin(\phi)^4 - 364 \sin(\phi)^2 + 7)$$

n=10, m=3

$$\frac{6435}{16} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} \sin(\phi) (323 \sin(\phi)^6 - 357 \sin(\phi)^4 + 105 \sin(\phi)^2 - 7)$$

$$n=10, m=4$$

$$\frac{45045}{16} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 (323 \sin(\phi)^6 - 255 \sin(\phi)^4 + 45 \sin(\phi)^2 - 1)$$

$$n=10, m=5$$

$$\frac{135135}{8} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} \sin(\phi) (323 \sin(\phi)^4 - 170 \sin(\phi)^2 + 15)$$

$$n=10, m=6$$

$$-\frac{675675}{8} (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 (323 \sin(\phi)^4 - 102 \sin(\phi)^2 + 3)$$

$$n=10, m=7$$

$$\frac{11486475}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(7/2)} \sin(\phi) (19 \sin(\phi)^2 - 3)$$

$$n=10, m=8$$

$$\frac{34459425}{2} (\sin(\phi) - 1)^4 (\sin(\phi) + 1)^4 (19 \sin(\phi)^2 - 1)$$

$$n=10, m=9$$

$$654729075 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(9/2)} \sin(\phi)$$

$$n=10, m=10$$

$$-654729075 (\sin(\phi) - 1)^5 (\sin(\phi) + 1)^5$$

$$n=11, m=0$$

$$\frac{1}{256} \sin(\phi)$$

$$(88179 \sin(\phi)^{10} - 230945 \sin(\phi)^8 + 218790 \sin(\phi)^6 - 90090 \sin(\phi)^4 + 15015 \sin(\phi)^2 - 693)$$

$$n=11, m=1$$

$$\frac{33}{256} \sqrt{-(\sin(\phi) - 1)(\sin(\phi) + 1)} \\ (29393 \sin(\phi)^{10} - 62985 \sin(\phi)^8 + 46410 \sin(\phi)^6 - 13650 \sin(\phi)^4 + 1365 \sin(\phi)^2 - 21)$$

$$n=11, m=2$$

$$-\frac{2145}{128} (\sin(\phi) - 1)(\sin(\phi) + 1) \sin(\phi) \\ (2261 \sin(\phi)^8 - 3876 \sin(\phi)^6 + 2142 \sin(\phi)^4 - 420 \sin(\phi)^2 + 21)$$

$$n=11, m=3$$

$$\frac{45045}{128} (-(\sin(\phi) - 1)(\sin(\phi) + 1))^{(3/2)} \\ (969 \sin(\phi)^8 - 1292 \sin(\phi)^6 + 510 \sin(\phi)^4 - 60 \sin(\phi)^2 + 1)$$

$$n=11, m=4$$

$$\frac{135135}{16} (\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 \sin(\phi) (323 \sin(\phi)^6 - 323 \sin(\phi)^4 + 85 \sin(\phi)^2 - 5)$$

$$n=11, m=5$$

$$\frac{135135}{16} (-(\sin(\phi) - 1)(\sin(\phi) + 1))^{(5/2)} (2261 \sin(\phi)^6 - 1615 \sin(\phi)^4 + 255 \sin(\phi)^2 - 5)$$

$$n=11, m=6$$

$$-\frac{2297295}{8} (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 \sin(\phi) (399 \sin(\phi)^4 - 190 \sin(\phi)^2 + 15)$$

$$n=11, m=7$$

$$\frac{34459425}{8} (-(\sin(\phi) - 1)(\sin(\phi) + 1))^{(7/2)} (133 \sin(\phi)^4 - 38 \sin(\phi)^2 + 1)$$

$$n=11, m=8$$

$$\frac{654729075}{2} (\sin(\phi) - 1)^4 (\sin(\phi) + 1)^4 \sin(\phi) (7 \sin(\phi)^2 - 1)$$

n=11, m=9

$$\frac{654729075}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(9/2)} (21 \sin(\phi)^2 - 1)$$

n=11, m=10

$$-13749310575 (\sin(\phi) - 1)^5 (\sin(\phi) + 1)^5 \sin(\phi)$$

n=11, m=11

$$13749310575 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(11/2)}$$

n=12, m=0

$$\frac{676039}{1024} \sin(\phi)^{12} - \frac{969969}{512} \sin(\phi)^{10} + \frac{2078505}{1024} \sin(\phi)^8 - \frac{255255}{256} \sin(\phi)^6 + \frac{225225}{1024} \sin(\phi)^4 - \frac{9009}{512} \sin(\phi)^2 + \frac{231}{1024}$$

n=12, m=1

$$\frac{39}{256} \sqrt{-(\sin(\phi) - 1) (\sin(\phi) + 1) \sin(\phi)} (52003 \sin(\phi)^{10} - 124355 \sin(\phi)^8 + 106590 \sin(\phi)^6 - 39270 \sin(\phi)^4 + 5775 \sin(\phi)^2 - 231)$$

n=12, m=2

$$-\frac{3003}{256} (\sin(\phi) - 1) (\sin(\phi) + 1) (7429 \sin(\phi)^{10} - 14535 \sin(\phi)^8 + 9690 \sin(\phi)^6 - 2550 \sin(\phi)^4 + 225 \sin(\phi)^2 - 3)$$

n=12, m=3

$$\frac{15015}{128} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(3/2)} \sin(\phi) (7429 \sin(\phi)^8 - 11628 \sin(\phi)^6 + 5814 \sin(\phi)^4 - 1020 \sin(\phi)^2 + 45)$$

n=12, m=4

$$\frac{135135}{128}$$

$$(\sin(\phi) - 1)^2 (\sin(\phi) + 1)^2 (7429 \sin(\phi)^8 - 9044 \sin(\phi)^6 + 3230 \sin(\phi)^4 - 340 \sin(\phi)^2 + 5)$$

$$n=12, m=5$$

$$\frac{2297295}{16}$$

$$(-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(5/2)} \sin(\phi) (437 \sin(\phi)^6 - 399 \sin(\phi)^4 + 95 \sin(\phi)^2 - 5)$$

$$n=12, m=6$$

$$-\frac{2297295}{16} (\sin(\phi) - 1)^3 (\sin(\phi) + 1)^3 (3059 \sin(\phi)^6 - 1995 \sin(\phi)^4 + 285 \sin(\phi)^2 - 5)$$

$$n=12, m=7$$

$$\frac{130945815}{8} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(7/2)} \sin(\phi) (161 \sin(\phi)^4 - 70 \sin(\phi)^2 + 5)$$

$$n=12, m=8$$

$$\frac{654729075}{8} (\sin(\phi) - 1)^4 (\sin(\phi) + 1)^4 (161 \sin(\phi)^4 - 42 \sin(\phi)^2 + 1)$$

$$n=12, m=9$$

$$\frac{4583103525}{2} (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(9/2)} \sin(\phi) (23 \sin(\phi)^2 - 3)$$

$$n=12, m=10$$

$$-\frac{13749310575}{2} (\sin(\phi) - 1)^5 (\sin(\phi) + 1)^5 (23 \sin(\phi)^2 - 1)$$

$$n=12, m=11$$

$$316234143225 (-(\sin(\phi) - 1) (\sin(\phi) + 1))^{(11/2)} \sin(\phi)$$

$$n=12, m=12$$

$$316234143225 (\sin(\phi) - 1)^6 (\sin(\phi) + 1)^6$$